THE INVERSE PROBLEM FOR A SYSTEM OF POROELASTICITY EQUATIONS: THE CASE WITH AN UNKNOWN DARCY TIME-DEPENDENT COEFFICIENT

Kholmatjon Imomnazarov (Institute of Computational Mathematics and Mathematical Geophysics, Russia),
Lochin Khujaaev (Tashkent University of Information Technologies (Karshi branch), Uzbekistan),
Zoyir Yangiboev (Karshi State University, Uzbekistan)

Abstract. This paper deals with studying the solvability of the inverse problem with an unknown Darcy time-dependent coefficient, for a system of poroelasticity equations, and with the uniqueness of its solution. The problem is considered in a rectangular domain. The conditions of the direct initial-boundary value problem and the integral redefinition condition necessary for finding the unknown coefficient are specified. In the proof of solvability, the parameter continuation, the fixed-point, the cut-off and the regularization methods are used.

Keywords. Inverse problem, systems of equations, Darcy coefficient, partial density, regularization, porous medium.

1. Introduction

In this paper, by the inverse problem for a system of poroelasticity equations we mean a task in which, together with obtaining the solution, it is necessary to determine the right-hand side (mass force) or and one or another coefficient (coefficients) of the equation itself. If the solution and the right-hand side are unknown in the inverse problem, then such an inverse problem will be linear; if the solution and, at least, one of the coefficients are unknown, then the inverse problem will be nonlinear. It is the nonlinear inverse problems for a system of poroelasticity equations that will be studied in this paper. The nonlinear inverse problems for hyperbolic equations in various statements were studied in [1–21] and in a number of others, however, in the statement below, similar problems have not been studied before.

2. Statement of the problem

Let $D$ be an interval $(0,1)$, $Q$ is the rectangle $\{(x,t): x \in D, t \in (0,T), 0 < T < +\infty\}$. Further, let $a(x)$, $f(x,t)$, $K(x,t)$, $u_0(x)$, $u_1(x)$ and $\mu(t)$ be the functions defined for $x \in [0,1]$, $t \in [0,T]$.

The inverse problem $A_*$, Find the functions $u(x,t)$, $v(x,t)$ and $s(t)$, in the rectangle $Q$ satisfying the system of equations [18-22]:

\[
\begin{align*}
\frac{u_n}{n} - u_{xx} + \gamma s(t) a(x)(u_n - v_n) &= f(x,t) \\
\frac{v_n}{n} - s(t) a(x)(u_n - v_n) &= f(x,t)
\end{align*}
\]

(1)

when the boundary conditions for the function $u(x,t)$ are valid:

$u(0,t) = u(1,t) = 0, \ 0 < t < T,$

(2)

and the initial conditions for the functions $u(x, t)$ and $v(x, t)$

\[
\begin{align*}
u(x,0) &= u_0(x), \ u_1(x,0) = u_1(x), \quad x \in D \\
v(x,0) &= 0, \ v_1(x,0) = 0, \quad x \in D
\end{align*}
\]

(3)

as well as the additional condition
\[
\int_0^1 K(x,t)u(x,t)\,dx = \mu(t), \quad 0 < t < T
\]  
(4)

are valid.

In system (1), \(u(x,t)\) and \(v(x,t)\) are the components of the displacement velocity of the elastic porous body, and the saturating fluid with the corresponding constant partial densities \(\rho_s\) and \(\rho_l\), for simplicity, we assume that the shear wave velocity is constant and equal to unity, \(\gamma = \rho_s / \rho_l\), and the function of the form \(s(t)a(x)\) characterizing the Darcy coefficient is responsible for the dissipation of energy in the system. For simplicity, we assume that \(a(1) = 1\).

Let \(V_0\) and \(V_1\) be the following spaces
\[
V_0 = \left\{v(x,t) : v(x,t) \in L_\infty\left(0,T; H^2(D) \cap H^1(0)\right)\right\},
\]
\[
v_1(x,t) \in L_\infty\left(0,T; H^1(D)\right), v_3(x,t) \in L_\infty\left(0,T; L_2(D)\right)\right\},
\]
\[
V_1 = \left\{v(x,t) : v(x,t) \in L_\infty\left(0,T; H^2(D) \cap H^1(0)\right)\right\},
\]
\[
v_4(x,t) \in L_\infty\left(0,T; H^1(D)\right), v_5(x,t) \in L_2(Q), v_{ext}(x,t) \in L_2(Q)\right\};
\]
the norm in these spaces is defined by the equalities
\[
\|v\|_{V_0} = \|v\|_{L_\infty(0,T; H^2(D))} + \|v_3\|_{L_\infty(0,T; H^1(D))} + \|v_2\|_{L_\infty(0,T; L_2(D))},
\]
\[
\|v\|_{V_1} = \|v\|_{L_\infty(0,T; H^1(D))} + \|v_3\|_{L_\infty(0,T; H^1(D))} + \|v_2\|_{L_2(Q)} + \|v_{ext}\|_{L_2(Q)}.
\]
Here \(H^1(D)\) denotes the Sobolev space [23].

For convenience, we introduce the following notation. Namely, we set
\[
F(t) = \int_0^1 K(x,t) f(x,t)\,dx,
\]
\[
M_1 = \rho \int_0^1 u_0^2(x)\,dx + \int_0^1 u_1^2(x)\,dx + \int_0^T f^2(x,t)\,dx\,dt \exp(T), \quad \rho = \rho_s + \rho_l,
\]
\[
M_2 = M_1T + \int_0^1 u_0^2(x)\,dx + \int_0^1 u_1^2(x)\,dx + \int_0^T f^2(x,t)\,dx\,dt,
\]
\[
M_3 = 2 \int_Q f_t^2(x,t)\,dx\,dt + 4vrai \max_{0 \leq t \leq T} \int_0^1 f^2(x,t)\,dx + \int_0^1 u_1^2(x)\,dx + 2 \int_0^1 u_0^2(x)\,dx - 4 \int_0^1 f(x,0)u_0^2(x)\,dx \exp(T),
\]
\[
M_4 = 2M_2T + \int_Q f_t^2(x,t)\,dx\,dt + 4vrai \max_{0 \leq t \leq T} \int_0^1 f^2(x,t)\,dx + \int_0^1 u_1^2(x)\,dx + 2 \int_0^1 u_0^2(x)\,dx - 4 \int_0^1 f(x,0)u_0^2(x)\,dx \exp(T),
\]
\[ N_0 = \max_{0 \leq x < T} \int_0^K (x,t) \, dx, \quad N_1 = \max_{0 \leq x < T} \int_0^K \phi(x,t) \, dx, \quad N_2 = \max_{0 \leq x < T} \int_0^K \phi^2(x,t) \, dx, \quad N_3 = N_1^2 \max_{0 \leq x < T} |a(t,1)|, \]
\[ N_4 = \int_0^T |a_t(x,t)| \left( \int_0^T \phi^2(y,t) \, dy \right) \, dx, \quad N_5 = \frac{N_0^2 + 2N_1^2 + N_2^2}{\gamma(m_1 - \bar{m}_1)}, \]
\[ N_6 = \frac{1}{\gamma(m_1 - \bar{m}_1)} \max_{0 \leq T} |F(t) - \mu^*(t)|, \quad \alpha = 2N_6 + N_5, \quad \beta = M'_1 + N_0T, \quad M'_1 = M'_1 + M_3, \]
\[ M'_1 = \int_0^T u_{x}^2(x) \, dx + \int_0^T u_{x}^2(x) \, dx + \int_0^T F^2(x,t) \, dx, \quad R_0 = 2 \left( N_1 M'_1 \right)^2 + \left( N_2 M'_2 \right)^2 + \left( N_3 M'_3 \right)^2, \]
\[ R'_3 = N_3 M'_2 + N_1 M'_3. \]

3. Investigation of the inverse problem \( A_x \).

**Theorem 1.** Let the following conditions be satisfied:
\[
\gamma \sqrt{\beta T} < 2, \quad a(x) \in C^1[0,1], \quad K(x,t) \in C^2(\overline{Q}), \quad \mu(t) \in C^2[0,T];
\]
\[
F(t) = \mu^*(t) \leq -m_0 < 0, \quad \mu'(t) \leq -m_1 < 0 \quad \text{at} \quad t \in [0,T], \quad R_0 \leq m_0, \quad R_1 \leq m_1;
\]
\[
\int_0^T K(x,0)u_0(x) \, dx = \mu(0), \quad \int_0^T K(x,0)u_t(x) \, dx + \int_0^T K(x,0)u_{x}(x) \, dx = \mu'(0).
\]

Then for any function \( f(x,t) \) such that \( f(x,t) \in L_2(Q) \), \( f_t(x,t) \in L_2(Q) \), and for any functions \( u_0(x) \) and \( u_t(x) \) such that \( u_0(x) \in H^2(D) \cap H^1(D) \) and \( u_t(x) \in H^1(D) \), the inverse problem \( A_x \) has a solution \( \{u(x,t), s(t)\} \) such that \( u(x,t) \in V_0, \quad v(x,t) \in L_\infty(0,T;L_2(Q)), \quad v_x(x,t) \in L_\infty(0,T;L_2(Q)), \quad v_{x_t}(x,t) \in L_\infty(0,T;L_2(Q)), \quad s(t) \in L_\infty[0,T]. \)

Proof. Let \( U(x,t), \, V(x,t) \) be given functions. Define the functions \( \phi(t,U), \, \psi(t,U,V) \) and \( s(t,U,V) \):
\[
\phi(t,U) = 2 \int_0^T K_x(x,t)U_x(x,t) \, dx + \int_0^T K_0(x,t)U(x,t) \, dx - K(0,t)U_x(0,t) + K(1,t)U_x(1,t) + \int_0^T K_{xx}(x,t)U(x,t) \, dx,
\]
\[
\psi(t,U,V) = \int_0^T K_x(x,t)U(x,t) \, dx + \int_0^T a_x(x) \left( \int_0^T K(y,t)U_y(y,t) \, dy \right) \, dx + \int_0^T a(x)K(x,t)V_x(x,t) \, dx,
\]
\[
s(t,U,V) = \frac{F(t) - \mu^*(t) + \phi(t,U)}{\gamma(\mu'(t) - \psi(t,U,V))}.
\]

Consider the initial-boundary value problem: find the functions \( u(x,t), \, v(x,t) \), which are the solution to the system of equation in the rectangle \( Q \):
such that conditions (2) and (3) are satisfied.

We show that this boundary value problem has the solution that belongs to the space $V_0$.

We use a combination of the cut-off, the regularization, and the fixed-point methods.

Let $\bar{m}_1$ be an arbitrary number from the interval $(R_1, m_1)$. Define the cutting functions $G_1(\xi)$ and $G_2(\xi)$:

$$G_1(\xi) = \begin{cases} \xi, & \text{if } |\xi| \leq \bar{a}, \\ m_0, & \text{if } |\xi| > m_0, \end{cases} \quad G_2(\xi) = \begin{cases} \xi, & \text{if } |\xi| \leq \bar{m}_1, \\ \bar{m}_1, & \text{if } |\xi| > \bar{m}_1. \end{cases}$$

Using the functions $G_1(\xi)$ and $G_2(\xi)$, we define the function $\mathcal{G}(t, U, V)$:

$$\mathcal{G}(t, U, V) = \frac{F(t) - \mu(t) + G_1(\phi(t, U))}{\gamma(\mu(t) - G_2(\psi(t, U, V)))}.$$ 

Let $\varepsilon_0$ be a fixed positive number, $\varepsilon$ is a positive number from the half-interval $(0, \varepsilon_0]$.

Consider the initial-boundary value problem: find the functions $u(x,t), v(x,t)$, which are the solution to the system of equations in the rectangle $Q$:

$$\begin{cases} u_t - u_{xx} + \gamma s(t, U, V) a(x)(u_t - v) = f(x, t) \\ v_t + \mathcal{G}(t, U, V) a(x)(u_t - v) = f(x, t) \end{cases} \quad (1'_e)$$

such that conditions (2) and (3) are satisfied.

It is for proving the solvability of the boundary value problem $(1'_e), (2), (3)$ (for fixed $\varepsilon$) that we use the fixed point method.

Let the function $f(x,t)$ belong to the space $L_2(Q), U(x,t), V(x,t)$ is an arbitrary function from the space $V_1$.

Consider the following linear initial-boundary value problem: find the functions $u(x,t), v(x,t)$, which are the solution to the system of equation, in the rectangle $Q$:

$$\begin{cases} u_t - u_{xx} + \gamma \mathcal{G}(t, U, V) a(x)(u_t - v) - \varepsilon u_{xx} = f(x, t) \\ v_t - \mathcal{G}(t, U, V) a(x)(u_t - v) = f(x, t) \end{cases} \quad (1'_{e,U,V})$$

such that conditions (2) and (3) are satisfied.

Solving the Cauchy problem $(1'_{e,U,V})$ with respect to $v(x, t)$ with zero initial conditions, we obtain

$$v_t(x, t) = \int_0^t [\mathcal{G}(\eta, U, V) a(x) u_\eta(x, \eta) + f(x, \eta)] \exp \left(-a(x) \int_\eta^t \mathcal{G}(\xi, U, V) d\xi \right) d\eta.$$ 

Further we will use this equality. In other words, we consider the initial-boundary-value problem $(1'_{e,U,V})$ with respect to the function $u(x,t)$, and denote the resulting integro-differential equation by $(1''_{e,U})$:

$$u_t - u_{xx} + \gamma \mathcal{G}(t, U, V) a(x)(u_t - u_{xx} -$$
\[-\gamma \overline{\varphi}(t, U) a(x) \int_0^t \tau (\eta, U) a(x) u_\eta (x, \eta) \exp \left\{ -a(x) \int_\eta^t \varphi (\xi, U) d\xi \right\} d\eta = f(x, t) + \]

\[\gamma \overline{\varphi}(t, U) a(x) \int_0^t \tau (\eta, U) a(x) f(x, \eta) \exp \left\{ -a(x) \int_\eta^t \varphi (\xi, U) d\xi \right\} d\eta , \]

Denote by \( \overline{\varphi} (t, u) = \mathcal{R}(t, U, v) \), where

\[v(x, \tau) = \int_0^{\tau - \eta} [\mathcal{R}(\eta, U, V) a(x) u_\eta (x, \eta) + f(x, \eta)] \exp \left\{ -a(x) \int_\eta^{\tau - \eta} \mathcal{R}(\xi, U, V) d\xi \right\} d\eta . (5') \]

Therefore, the initial-boundary value problem is the first initial-boundary value problem for equations of a composite type with an integral term (also referred to in some published works as pseudo-hyperbolic, or the Sobolev type equations). The solvability of this initial-boundary-value problem in the space \( f(x, t) \) (if the function \( f(x, t) \) belongs to the space \( L_2(Q) \)) is established using the methods, for example, as in [24, 25]. Since the function \( \mathcal{R}(x, t) \) also belongs to the space \( V_1 \), we obtain that the initial-boundary value problem \((V_1)\), (2), (3) generates the operator \( \Phi \), which takes the space \( V_1 \) into itself: \( \Phi (\mathcal{R}) = u \). We show that the operator \( \Phi \) has fixed points in the space \( V_1 \). We will do this using the Schauder theorem.

Let us consider the equality

\[\int_0^t \left[ u_{\tau \tau} - u_{xx} + \gamma \mathcal{R}(t, U, V) (u_{\tau - \eta} - \varepsilon u_{\tau \tau}) \right] u_{\tau} dx d\tau = \int_0^t f_{\tau} dx d\tau , \]

\[\int_0^t \left[ v_{\tau \tau} - \mathcal{R}(t, U, V) (u_{\tau - \eta} - \varepsilon u_{\tau \tau}) \right] v_{\tau} dx d\tau = \int_0^t f_{\tau} dx d\tau \]

resulting from system \((1', \mathcal{R}, V)\). From this equality it is easy to pass to the inequality

\[\int_0^t u_{\tau}^2 (x, t) dx + \int_0^t u_{\tau \tau}^2 (x, t) dx + 2\gamma \int_0^t \mathcal{R}(\tau, U, V) (u_{\tau \tau}^2 (x, \tau) - u_{\tau} (x, \tau) v_{\tau} (x, \tau)) d\tau + \]

\[+ 2\varepsilon \int_0^t u_{\tau \tau}^2 (x, \tau) d\tau \leq \int_0^t u_{\tau}^2 (x, t) dx d\tau + \int_0^t \left[ u_{\tau}^2 (x, \tau) + u_{\tau \tau}^2 (x, \tau) \right] dx + \int_0^t f_{\tau}^2 (x, t) d\tau , \]

\[\int_0^t v_{\tau}^2 (x, t) dx - 2\gamma \int_0^t \mathcal{R}(\tau, U, V) (u_{\tau} (x, \tau) v_{\tau} (x, \tau) - v_{\tau \tau}^2 (x, \tau)) d\tau \leq \]

\[\leq \int_0^t v_{\tau}^2 (x, \tau) d\tau + \int_0^t f_{\tau}^2 (x, t) d\tau . \]

Multiplying the first equation by \( \rho_0 \), and the second one by \( \rho_1 \), after some transformations we come to

\[\rho_0 \int_0^t u_{\tau}^2 (x, t) dx + \rho_1 \int_0^t v_{\tau}^2 (x, t) dx + \rho_2 \int_0^t u_{\tau}^2 (x, t) dx + \]

\[+ 2\varepsilon \int_0^t \mathcal{R}(\tau, U, V) (u_{\tau} (x, \tau) - v_{\tau} (x, \tau))^2 d\tau + 2\varepsilon \rho_2 \int_0^t u_{\tau \tau}^2 (x, \tau) d\tau \leq \]
\[
\begin{align*}
&= \int_0^t \int_0^1 \rho_1 u^2_{\tau}(x,t) dx d\tau + \int_0^t \int_0^1 \rho_2 v^2_{\tau}(x,t) dx d\tau + \rho_1 \int_0^t \left[ u_{\tau}^2(x) + u^2(x) \right] dx + \rho \int_0^t f^2(x,t) dx d\tau. \\
&\leq \rho_1 \int_0^t u^2_{\tau}(x,t) dx + \rho_2 \int_0^t v^2_{\tau}(x,t) dx + \rho \int_0^t f^2(x,t) dx d\tau (6)
\end{align*}
\]

Further, bearing in mind that due to the conditions of the theorem and according to its construction, the function \( \mathcal{K}(\tau, U, V) \) is non-negative, and using the Gronwall lemma, we obtain the estimate

\[
\begin{align*}
\rho_1 \int_0^t u^2_{\tau}(x,t) dx + \rho_2 \int_0^t v^2_{\tau}(x,t) dx &\leq \frac{1}{\rho_1} \exp(T) = M_1.
\end{align*}
\]

From inequality (6) and estimate (7) follows

\[
\int_0^t \left[ u^2_{\tau}(x,t) + u^2(x,t) \right] dx + \epsilon \int_0^t u^2_{\tau}(x,t) dx d\tau \leq M_2 / \rho_\tau.
\]

Therefore, from inequalities (7) and (8) we have

\[
\int_0^t u^2_{\tau}(x,t) dx \leq M_2 / \rho_\tau, \quad \int_0^t v^2_{\tau}(x,t) dx \leq M_1 / \rho_\tau.
\]

Now consider the equality

\[
\begin{align*}
\int_0^t \left[ u^2_{\tau} - u^2_{xx} + \mathcal{K}(\tau, U, V) u_{\tau} - \epsilon u_{xx} - \mathcal{K}(\tau, U, V) \right] \times \left[ \mathcal{K}(\eta, U, V) a(x) u_{\eta}(x, \eta) + f(x, \eta) \right] \\
\times \exp \left[ -a(x) \int_\eta^\tau \mathcal{K}(\xi, U, V) d\xi \right] d\eta
\end{align*}
\]

\[
\begin{align*}
\int_0^t u^2_{\tau} d\tau = \int_0^t f u_{\tau} dx d\tau,
\end{align*}
\]

that is also a consequence of equation (1'\( \tau, U \)).

Now consider the integral

\[
J = \int_0^t \int_0^1 \mathcal{K}(\tau, U, V) \left[ \mathcal{K}(\eta, U, V) a(x) u_{\eta}(x, \eta) + f(x, \eta) \right] \exp \left[ -a(x) \int_\eta^\tau \mathcal{K}(\xi, U, V) d\xi \right] d\eta dx d\tau.
\]

Thus, we have

\[
|J| \leq \max_{[0,1]} a(x) \int_0^t \int_0^1 \mathcal{K}(\tau, U, V) \left[ \mathcal{K}(\eta, U, V) u_{\eta}(x, \eta) \right] d\eta dx d\tau + \int_0^t \int_0^1 \mathcal{K}(\tau, U, V) f(x, \eta) d\eta dx d\tau.
\]

Hence, taking into account the inequalities [15]

\[
|\nu_{\tau}(0, t)| \leq \left( \int_0^t |\nu_{xx}(x, t)|^2 d\tau \right)^{1/2},
\]

\[
|\nu_{\tau}(1, t)| \leq \left( \int_0^t |\nu_{xx}(x, t)|^2 d\tau \right)^{1/2},
\]

\[
|a(1, t) \mu'(1) - G_2(\xi)| \geq m_1 - \tilde{m}_1,
\]

\[
|\phi(t, U)| \leq 2 \int_0^1 \left| K_1(x, t) U_1(x, t) \right| dx + \int_0^1 \left| K_2(x, t) U_1(x, t) \right| dx +
\]

\[
+ \left| K(0, t) U_1(0, t) \right| + \left| K(1, t) U_1(1, t) \right| + \int_0^1 \left| K_2(x, t) U_1(x, t) \right| dx \leq 2 \left( \int_0^1 \left| K_1(x, t) \right|^2 dx \right)^{1/2} \left( \int_0^1 \left| U_1(x, t) \right|^2 dx \right)^{1/2} +
\]
we obtain
\[
\|\mathcal{F}(\tau, U, V)\| \leq N_6 + N_5 \left( \int_0^1 \left[ U_{xx}^2(x,t) + U_{xx}^2(x,t) + U_{xx}^2(x,t) \right] dx \right)^{1/2} \tag{13}
\]
Integrating equality (9) in its left-hand side by parts and taking into account the estimates, obtained, we come to the third estimate after conducting simple transformations
\[
\int_0^1 u_{xx}^2(x,t) dx + \int_0^1 u_{xx}^2(x,t) dx + \int_0^1 u_{xx}^2(x,t) dx + \int_0^1 \mathcal{F}(\tau, U, V) u_{xx}^2(x,t) dx d\tau + e \int_0^1 u_{xx}^2(x,t) dx d\tau \leq \\
+N_5 \left( \int_0^1 \left[ u_{xx}^2(x,t) + u_{xx}^2(x,t) + u_{xx}^2(x,t) \right] dx \right)^{3/2} + C_1, \tag{14}
\]
where
\[
C_1 = \int_0^1 \left[ u_{xx}^2(x) + u_{xx}^2(x) \right] dx + \int_0^1 \int_0^1 f^2(x,t) dx d\tau.
\]
Combining inequalities (8) and (14), we arrive at the inequality
\[
y(t) \leq N_6 \int_0^1 y(\tau) d\tau + N_5 \int_0^1 \frac{3}{2} y^2(\tau) d\tau + M'. \tag{15}
\]
In formula (15): \( y(t) = \int_0^1 \left[ u_{xx}^2(x,t) + u_{xx}^2(x,t) + u_{xx}^2(x,t) \right] dx \).

The following inequality \( 3y(t) \leq 2y^{3/2}(t) + 1 \) makes it possible to simplify inequality (15) and to pass to the inequality
\[
y(t) \leq \alpha \int_0^1 \frac{3}{2} y^2(\tau) d\tau + \beta.
\]
Properties of the solutions to the integral inequalities [26, Ch. III] make possible to obtain for the function \( y(t) \) the estimate \( y(t) \leq y_0(t) \), where the function \( y_0(t) \) is defined as the solution to the Cauchy problem \( y_0'(t) = \alpha y_0^3(t), \quad y_0(0) = \beta \).

The function \( y_0(t) \) has the form \( y_0(t) = \frac{4\beta}{(2-\alpha \sqrt{\beta t})^2} \). Consequently, for the solutions of the initial-boundary value problem 1 (1xU, (2), (3), an an a priori estimate holds:
\[
y(t) \leq \frac{4\beta}{(2-\alpha \sqrt{\beta T})^2}. \tag{16}
\]
From inequalities (14) and (16) we obtain
\[
\int_0^1 u_{xx}^2(x,t) dx + \int_0^1 u_{xx}^2(x,t) dx + \int_0^1 \mathcal{F}(\tau, U, V) u_{xx}^2(x,t) dx d\tau + e \int_0^1 u_{xx}^2(x,t) dx d\tau \leq C' \tag{17}
\]
If now we consider the equality
\[
\int_0^1 \left[ u_{xx}^2 + u_{xx}^2 + \mathcal{F}(\tau, U, V) u_{xx}^2 + e u_{xx}^2 - \mathcal{F}(\tau, U, V) \times
\]

ISSN 2521-3261 (Online)/ ISSN 2521-3253 (Print)
DOI 10.37057/2521-3261 https://journalofresearch.info/
\[
\times \int_0^1 \left[ \mathcal{K}(\eta, U, V) a(x) u_\eta(x, \eta) + f(x, \eta) \right] \exp \left( -a(x) \int_0^\eta \mathcal{K}(\xi, U, V) d\xi \right) d\eta \] 
\[
\int_0^1 u^2(x, t) dx dt \leq C_2
\]
then with allowance for estimate (17) and the form of the function \( \mathcal{K}(t, U, V) \), it is easy to obtain the following estimate

\[
\int_0^1 u^2(x, t) dx dt \leq C_2
\]

with the constant \( C_2 \), determined only by the norm of the function \( f(x, t) \) in the space \( L_2(Q) \), and by the number \( C_1 \).

From estimates (16), (17), and (18), the resulting a priori estimate is derived for the initial-boundary value problem solutions (1*), (2), (3)

\[
\|u\|_{V_1} \leq C_0
\]

with the constant \( C_0 \), defined only by the numbers \( M_2, C_1 \) and \( C_2 \).

With the help of estimate (19) we will prove that all the conditions of the Schauder theorem are fulfilled for the operator \( \Phi \).

First of all, note that from estimate (19) it follows that the operator \( \Phi \) transfers the ball of radius \( C_0 \) of the space \( V_1 \) into itself.

Let us prove that the operator \( \Phi \) is continuous.

Let the sequence of functions \( \{v_m(x, t)\} \) converge in the space \( V_1 \) to the function, \( v(x, t) \), \( u_m(x, t) \) and \( u(x, t) \) are images of the functions \( v_m(x, t) \) and \( v(x, t) \) and with the operator action, \( \Phi \), \( w_m(x, t) \) are the functions \( u_m(x, t) - u(x, t) \).

The functions \( w_m(x, t) \) are the solutions to the initial-boundary value problem

\[
w_{max'} - w_{max} + \gamma \mathcal{K}(t, u, v) w_{max} - \varepsilon w_{max} =
\]

\[
-\mathcal{K}(t, u, v) \int_0^1 \left[ \mathcal{K}(\eta, u, v) a(x) (u_\eta - u_{\eta max}) + \left( \mathcal{K}(\eta, u, v) - \mathcal{K}(\eta, u_{max}, v_{max}) \right) (u_{\eta max} - 
\]

\[
-\int_0^1 \left[ \mathcal{K}(\xi, u_{max}, v_{max}) a(x) u_{\xi max}(x, \xi) + f(x, \xi) \right] \exp \left( -a(x) \int_\xi^1 \mathcal{K}(\xi, u_{max}, v_{max}) d\xi \right) d\xi 
\]

\[
\times \exp \left( -a(x) \int_\eta^1 \mathcal{K}(\xi, U, V) d\xi \right) d\eta = \gamma (\mathcal{K}(t, u, v) - \mathcal{K}(t, u_{max}, v_{max})) u_{max},
\]

\[
w_m(0, t) = w_m(1, t) = 0, \ 0 < t < T, \ w_m(x, 0) = w_{max}(x, 0) = 0, \ x \in D.
\]

By repeating the proof of estimate (19), it is not difficult to make sure that the functions \( w_m(x, t) \) can be evaluated:

\[
\|w_m\|_{V_1} \leq C_0 \|\mathcal{K}(t, u, v) - \mathcal{K}(t, u_{max}, v_{max})\|_{V_1(Q)}.
\]

The following equality takes place:

\[
\mathcal{K}(t, u, v) - \mathcal{K}(t, u_{max}, v_{max}) = \frac{1}{\left[ \mu'(t) - G_2(\psi(t, u, v)) \right] \left[ \mu'(t) - G_2(\psi(t, u_{max}, v_{max})) \right]} 
\]
\[
\times\left[(F(t) - \mu''(t))\left[G_2(\psi(t,u,v)) - G_2(\psi(t,u_m,v_m))\right] + \mu'(t)\left[G_1(\phi(t,u)) - G_1(\phi(t,u_m))\right] + \\
+ G_2(\psi(t,u,v))\left[G_1(\phi(t,u)) - G_1(\phi(t,u_m))\right] + \\
+ G_1(\phi(t,u))\left[G_2(\psi(t,u,v)) - G_2(\psi(t,u_m,v_m))\right]\right].
\]

Using inequality (10), the boundedness of the functions \(\mu'(t), \mu''(t), G_1(\xi), G_2(\xi)\), the Lipschitzness of the functions \(G_i(\xi)\) and \(G_j(\xi)\), as well as the belonging of the function \(F(t)\) to the space \(L_2[0, T]\), bring about the inequality

\[
\|\mathcal{G}(t,u,v) - \mathcal{G}(t,u_m,v_m)\|_{L_2(Q)} \leq C_1 \left[\|u - u_m\|_{L_2[0,T]} + \right] + \\
\|u_m - u\|_{L_2[0,T]} + \|u_{mx}(0,t) - u_x(0,t)\|_{L_2[0,T]} + \|u_{mx}(1,t) - u_x(1,t)\|_{L_2[0,T]}\]
\]

with the constant \(C_1\), defining the functions \(f(x,t), K(x,t), \mu(t)\), and the numbers \(m_0\) and \(m_1\).

Inequalities (15) and (16) with allowance for (10), (11) give the estimate

\[
\|w_m\|_{V_1} \leq C_1 \|v_m - v\|_{V_1}.
\]

This estimate means that the operator \(\Phi\) is continuous in the space \(V_1\).

Let us now prove that the operator \(\Phi\) is compact in the space \(V_1\).

Let \(\{v_m(x,t)\}\) be the boundedness of the sequence of the functions from the space \(V_1\), \(\{u_m(x,t)\}\) is the sequence of images of the functions \(u_m(x,t)\) with the operator action \(\Phi\). It follows from estimate (19) that the sequence \(\{u_m(x,t)\}\) is also bounded in the space \(V_1\). From the boundedness in the space \(V_1\) of the sequences \(\{v_m(x,t)\}\) and \(\{u_m(x,t)\}\), as well as from the theorem of complete continuity of imbeddings \(W^2_k(Q) \rightarrow W^1_k(Q)\), \(W^1_k(Q) \rightarrow L_2(\partial Q)\), and from the possibility of choosing a strongly convergent sub-sequence converging almost everywhere [23, 27] it follows that there are sub-sequences \(\{v_{m_k}(x,t)\}\) and \(\{u_{m_k}(x,t)\}\), as well as the functions \(v(x,t)\) and \(u(x,t)\) such that \(k \rightarrow \infty\), and there is the convergence \(v_{m_k}(x,t) \rightarrow v(x,t), u_{m_k}(x,t) \rightarrow u(x,t)\), which is weak in the space \(W^2_k(Q)\), \(v_{m_k}(x,t) \rightarrow v(x,t), v_{m_j}(x,t) \rightarrow v_j(x,t)\), almost everywhere in \(Q\), \(v_{m_k}(0,t) \rightarrow v(0,t), v_{m_k}(1,t) \rightarrow v(1,t)\), and almost everywhere on the segment \([0, T]\), \(u_{m_k} \rightarrow u(x,t)\), which is weak in the space \(L_2(Q)\). These convergences and continuity of the functions \(G_1(\xi)\) and \(G_2(\xi)\) mean, in particular, that the convergence \(\bar{u}(t,v_{m_k}) \rightarrow \bar{u}(t,v)\) occurs almost everywhere on the segment \([0,T]\). Obviously, the limit functions \(u(x,t)\) and \(v(x,t)\) in the rectangle \(Q\) satisfy equation \((V^*_{q,\ell})\). Let us set \(w_k(x,t) = u_{m_k}(x,t) - u(x,t)\). Since for the sequence \(\{w_k(x,t)\}\) estimate (22) holds, this sequence will converge in the space \(V_1\) to the identity of the zero function. In other words, we have proved that for any sequence \(\{v_m(x,t)\}\) bounded in the space \(V_1\), a strongly convergent sub-sequence \(\{\Phi v_m(x,t)\}\) can be extracted from the sequence in question. This means that the operator \(\Phi\) is compact in the space \(V_1\).
So, the operator $\Phi$ transfers the ball of the space $V_1$ of radius $C_0$ into itself and is quite continuous. Consequently, all the conditions of the Schauder theorem are met for this operator. According to this theorem, there exists a function $u(x,t)$ belonging to the space $V_1$ and being the solution to the initial-boundary value problem \((1^*_{\varepsilon,\nu}), (2), (3)\).

It will be established that for the solutions of initial-boundary value problem \((1'_{\varepsilon}), (2), (3)\) there are a priori estimates, which are uniform in terms of the parameter $\varepsilon$.

Now let for the function $f(x,t)$ the conditions of the theorem be valid. Obviously, estimates (7) and (8) are stored for initial-boundary value problem solutions \((1^*_{\varepsilon,\nu}), (2), (3)\). Further, if, in equation (9) corresponding to equation \((1^*_{\varepsilon,\nu})\), the partial integration is not only on the left-hand, but also on the right-hand side, and then on the right-hand side the Jung inequality is applied, it is not difficult to obtain the inequality

\[
\begin{align*}
\frac{1}{2} \int_0^1 u_{tt}^2(x,t) dx + \frac{1}{2} \int_0^1 u_{xx}^2(x,t) dx + & \int_0^1 \mathcal{G}(\tau, u) u_{tt}^2(x,t) dx d\tau + \varepsilon \int_0^1 u_{xx}^2(x,t) dx d\tau \\
\leq & \mathcal{N}_e \left( \int_0^1 u_t^2(x,t) dx + u_s^2(x,t) + u_{xx}^2(x,t) \right) \right\} dx + \\
& + \frac{1}{2} \int_0^1 u_{xx}^2(x,t) dx d\tau + \frac{1}{2} \int_0^1 f_t^2(x,t) dx d\tau + \varepsilon \int_0^1 u_{xx}^2(x,t) dx d\tau - \\
& - \frac{1}{2} f(x,0) u_0^2(x) dx + \frac{\delta^2}{2} \int_0^1 u_{xx}^2(x,t) dx + \frac{1}{2 \delta^2} \int f^2(x,t) dx,
\end{align*}
\]

where $\delta$ has an arbitrary positive number. Let us set $\delta = \frac{1}{\sqrt{2}}$, and we obtain that for the solutions of the initial-boundary value problem \((1^*_{\varepsilon,\nu}), (2), (3)\), the following inequality is valid:

\[
\begin{align*}
\int_0^1 u_{xx}^2(x,t) dx + & \int_0^1 u_{xx}^2(x,t) dx + \varepsilon \int_0^1 u_{xx}^2(x,t) dx d\tau \\
+ & 2 \int_0^1 f_t^2(x,t) dx d\tau + 4 \operatorname{vrai} \max_{0 \leq \tau \leq T} \int f^2(x,t) dx + \\
\leq & \int_0^1 u_t^2(x,t) dx + 2 \int_0^1 u_0^2(x) dx - 4 \int f(x,0) u_0^2(x) dx + C. \quad (23)
\end{align*}
\]

In this case we made use of inequality (16). From inequality (23) with the help of the Gronwall lemma, it is easy to derive a priori estimate

\[
\int_0^1 u_{xx}^2(x,t) dx \leq M_3. \quad (24)
\]

This estimate as well as inequality (23) mean that for the solutions of the initial-boundary value problem \((1'_{\varepsilon,\nu}), (2), (3)\), the following uniform estimate is performed

\[
\begin{align*}
\int_0^1 u_{xx}^2(x,t) dx + & \int_0^1 u_{xx}^2(x,t) dx + \varepsilon \int_0^1 u_{xx}^2(x,t) dx d\tau \\
\leq & \int_0^1 u_t^2(x,t) dx + 2 \int_0^1 u_0^2(x) dx - 4 \int f(x,0) u_0^2(x) dx + C. \quad (25)
\end{align*}
\]
Estimates (7), (8) and (25) show that there are inequalities for the initial-boundary value problem solutions (1′w,x,u), (2), (3)

\[ \phi(t,u) \leq 2 \left( N_1 M_1 \right)^{\frac{1}{2}} + \left( N_2 M_2 \right)^{\frac{1}{2}} + \left( N_0 M_0 \right)^{\frac{1}{2}} = R_0, \]  
(26)

\[ \psi(t,u,v) \leq N_3 M_3^{\frac{1}{2}} + N_4 M_4^{\frac{1}{2}} = R_1. \]  
(27)

The conditions of Theorem 1 and inequalities (26) and (27) mean that for the solutions to the initial-boundary value problem \( u(x,t), v(x,t), (1′w,x,u), (2), (3) \) the equalities hold:

\[ G_1(\phi(t,u)) = \phi(t,u), \quad G_2(\psi(t,u,v)) = \psi(t,u,v), \quad \delta(t,u,v) = s(t,u,v), \]

where the function \( v(x,t) \) is defined by formula (5′).

Obviously, the function \( s(t,u,v) \) is non-negative and bounded by the function from above on the segment \([0,T]\). Further, clearly, for the solutions to the initial-boundary value problem (1′w,x,u), (2), (3), an a priori estimate will be obtained:

\[ \int_{0}^{1} \left[ u_{\tau\tau}^2 (x,t) dx d\tau \right] \leq M_5 \]  
(28)

with the constant \( M_5 \) defined the numbers \( m_0, m_1, R_0, R_1, M_1 - M_4 \), and the functions \( K(x,t) \) and \( f(x,t) \).

Thus, we have proved that the initial-boundary value problem (1′w,x,u), (2), (3) has the solution \( u^\varepsilon(x,t) \) which is the solution of the initial-boundary value problem (1w,x), (2), (3) and such that for a family of functions \( \{u^\varepsilon(x,t)\} \) the uniform in \( \varepsilon \) a priori estimates (8), (23), (24) and (28) are valid. From these estimates and also, from the theorems of complete continuity of imbeddings \( W_{2}^2(Q) \rightarrow W_{1}^1(Q) \), \( W_{2}^1(Q) \rightarrow L_{2}(\partial Q) \) and of the possibility of choosing a strongly converging sequence of the subsequence almost everywhere, it follows that there are sequences \( \{\varepsilon_m\} \) of positive numbers and the function \( u(x,t) \) such that at \( m \rightarrow \infty \), the convergence takes place:

\( \varepsilon_m \rightarrow 0, \quad u_{\varepsilon_m} (x,t) \rightarrow u(x,t), \) which is weak in the space \( W_{2}^2(Q) \), \( \varepsilon_m (x,t) \rightarrow u(x,t), \) \( u_{\varepsilon_m} (x,t) \rightarrow u(x,t) \) almost everywhere in \( Q \), \( u_{\varepsilon_m} (0,t) \rightarrow u_{\varepsilon_m} (0,t), \) \( u_{\varepsilon_m} (1,t) \rightarrow u_{\varepsilon_m} (1,t) \), almost everywhere on the segment \([0,T]\), \( \varepsilon_m u_{\varepsilon_m}^\varepsilon (x,t) \rightarrow 0 \) is weak in the space \( L_{2}(Q) \). It follows from these convergences that almost everywhere on the segment \([0,T]\), and further that equation (1′) will be fulfilled for the limit function \( u(x,t) \). In this equation, the second and the third terms belong to the space \( L_{w_{6}}(0,T;L_{2}(D)) \) following from estimates (8), (23), and (24). Since the right-hand side in equation (1′) also belongs to the space \( L_{w_{6}}(0,T;L_{2}(D)) \), we obtain that the function \( u_{\varepsilon_m} (x,t) \) will also belong to the space \( L_{w_{6}}(0,T;L_{2}(D)) \). Then the function \( u(x,t) \) will belong to the space \( V_{0} \).
So, it is proven that the initial-boundary value problem (1'), (2), (3) has the solution \( u(x,t) \) belonging to the space \( V_0 \). Let us show that this solution and the function \( s(t) \) defined by the equality \( s(t) = s(t,u,v) \) will give the desired solution to the inverse problem \( A_j \).

Let us multiply equation (1') by the function \( K(x,t) \) and then integrate it along the segment \([0,1]\). Hence, we have the equality

\[
\frac{\partial^2}{\partial t^2} \left( \int_0^1 K(x,t)u(x,t)dx \right) - 2\int_0^1 K_x(x,t)u_t(x,t)dx - \int_0^1 K_t(x,t)u(x,t)dx - \int_0^1 K_{xx}(x,t)u(x,t)dx + s(t)\frac{\partial}{\partial t}\left( \int_0^1 K(x,t)u(x,t)dx \right) - \int_0^1 K_x(x,t)u(x,t)dx - \int_0^1 a_s(x)\left( \int_0^1 K(y,t)u(y,t)dy \right)dx = F(t).
\]

Further, this equality is derived directly from the representation of the function

\[
\mu'(t) - \phi(t,u) + s(t) [\mu'(t) - \psi(t, u, v)] = F(t).
\]

Subtracting from one of these equations the other one, we obtain the equality

\[
\frac{\partial^2}{\partial t^2} \left[ \int_0^1 K(x,t)u(x,t)dx - \mu(t) \right] + s(t)\frac{\partial}{\partial t} \left[ \int_0^1 K(x,t)u(x,t)dx - \mu(t) \right] = 0
\]

Let us designate

\[
\nu(t) = \int_0^1 K(x,t)u(x,t)dx - \mu(t).
\]

The consequence of equality (29) is the equality

\[
\frac{1}{2}\left[ \nu^2(t) - \nu^2(0) \right] + \int_0^t s(\tau)\nu^2(\tau)d\tau = 0.
\]

It follows from the conditions of agreement of Theorem 1 that there are equalities \( \nu = \nu'(0) = 0 \). Since the function \( s(t) \) is non-negative, equation (30) means that the function \( \nu'(t) \) is identical to the zero function. But then the function \( \nu(t) \) as it is, will be equal to zero owing to the equality \( \nu(0) = 0 \). The identical transformation of the function \( \nu(t) \) to the zero function means that the solution \( u(x,t) \) to the initial-boundary value problem (1'), (2), (3) satisfies condition (4). Together with the belonging of the functions \( u(x,t) \) and \( s(t) \) to the required classes (which in the course of the proof of Theorem 1 is shown) all this means that the functions \( u(x,t) \) and \( s(t) \) will give the desired solution to the inverse problem \( A_j \). The theorem is proved.

We will now discuss the uniqueness of solutions to the inverse problem \( A_j \).

Let \( W_1 \) be the set:

\[
W_1 = \{ u(x,t), v(x,t), s(t) : u(x,t) \in V_0, v(x,t) \in L_\infty(0, T; L^2(Q)), v_t(x,t) \in L_\infty(0, T; L^2(Q)), v_{tt}(x,t) \in L^2(Q), s(t) \in L_\infty[0,T], \mu'(t) - \psi(t,u,v) \geq k_0 > 0, s(t) \geq 0 \text{ at } t \in [0,T] \}.
\]
Theorem 2. Let for the functions, $a(x,t)$, $f(x,t)$, $K(x,t)$ and $\mu(t)$ the conditions of Theorem 1 be fulfilled. Then, in the set $W_i$, the inverse problem $A_i$ cannot have more than one solution.

Proof. Let $\{u_1(x,t), v_1(x,t), s_1(t)\}$ and $\{u_2(x,t), v_2(x,t), s_2(t)\}$ be two solutions to the inverse problem $A_i$ belonging to the set $W_i$. Let us set

$$w_1(x,t) = u_1(x,t) - u_2(x,t), \quad w_2(x,t) = v_1(x,t) - v_2(x,t),$$

$$f_1(x,t) = \gamma(s_2(t) - s_1(t))a(x)(u_2(t,x) - v_2(t,x)), \quad f_2(x,t) = [s_2(t) - s_1(t)]a(x)(u_2(t,x) - v_2(t,x)).$$

The following equalities take place

$$w_{1x} - w_{1xx} + 2\gamma s_1(t) a(x) w_{1t} = f_1(x,t) +$$

$$+ \gamma s_1(t) \int_0^t [s_1(\eta) a(x) w_{1\eta} + f_2(x,\eta)] \exp \left(-a(x) \int_\eta^t s_1(\xi) d\xi \right) d\eta,$$

$$w_1(x,0) = w_{1t}(x,0) = 0, \quad x \in D, \quad w_1(0,t) = w_1(1,t) = 0, \quad 0 < t < T.$$

The function $w_2(x,t)$ is defined by the function $w_1(x,t)$ according to the formula

$$w_2(x,t) = \int_0^t (t-\eta) \left[s_1(\eta) a(x) w_{1\eta} + f_2(x,\eta)\right] \exp \left(-a(x) \int_\eta^t s_1(\xi) d\xi \right) d\eta.$$

Let us consider the equality

$$\int_0^t \left[w_{1x} - w_{1xx} + s_1(t) a(x) w_{1t}\right] \left[w_{1x} - \left(x - \frac{1}{2}\right) w_{1x}\right] dx d\tau =$$

$$= \int_0^t f_1 \left[w_{1x} - \left(x - \frac{1}{2}\right) w_{1x}\right] dx d\tau + \gamma \int_0^t s_1(t) \int_0^t [s_1(\eta) a(x) w_{1\eta} + f_2(x,\eta)] \times$$

$$\times \exp \left(-a(x) \int_\eta^t s_1(\xi) d\xi \right) d\eta \left[w_{1x} - \left(x - \frac{1}{2}\right) w_{1x}\right] dx d\tau.$$

By integrating on the left by parts and applying the Jung inequality, we will come to the inequality

$$\int_0^1 \left[w^2_1(x,t) + w^2_2(x,t)\right] dx + \int_0^t \left[w^2_{1\eta}(0,\tau) + w^2_{1\eta}(1,\tau)\right] d\tau \leq$$

$$\gamma \int_0^t \int_0^t s_1(\tau) \left[s_1(\eta) a(x) w_{1\eta} + f_2(x,\eta)\right] d\eta \left[w_{1x} - \left(x - \frac{1}{2}\right) w_{1x}\right] dx d\tau +$$

$$+ \delta \int_0^t f_1^2 dx d\tau + C(\delta) \int_0^t (w^2_{1\eta} + w^2_{1\eta}) dx d\tau,$$

in which the number $\delta$ is an arbitrary positive number. The second integral is also evaluated as above through the function $y_0(t)$. Given the representations of the functions $s_1(t)$ and $s_2(t)$ and through the functions $u_1(x,t)$ and $u_2(x,t)$, respectively, and further considering the belonging of the solutions $\{u_1(x,t), s_1(t)\}$ and $\{u_2(x,t), s_2(t)\}$ to the set $W_i$, it is not difficult to obtain the inequality

$$\int_0^1 \int_0^t f^2_1 dx d\tau \leq M_0 \left[\int_0^1 \left(w^2_{1\eta} + w^2_{1\eta}\right) dx d\tau + \int_0^t \left[w^2_{1\eta}(0,\tau) + w^2_{1\eta}(1,\tau)\right] d\tau\right].$$

(32)
with the constant $M_0$ determined by the functions $a(x), f(x,t), K(x,t)$ and $\mu(t)$. Combining inequalities (31) and (32), then choosing a number $\delta$ to be sufficiently small, we obtain the inequality

$$
\int_0^1 \left[ w_{\tau}^2(x,t) + w_{\tau}^2(x,t) \right] dx \leq M'_0 \int_0^1 (w_{\tau}^2 + w_{\tau}^2) dx + M'_0.
$$

From this inequality and from the Gronwall lemma follows the identity $w_1(x,t) \equiv \text{const}$. Obviously, this identity is actually fulfilled as $w_2(x,t) \equiv 0$. The latter identity means that the functions $u_1(x,t)$ and $u_2(x,t)$ coincide in the rectangle $Q$.

From the coincidence of the functions $u_1(x,t)$ and $u_2(x,t)$ follows the coincidence of the functions $s_1(t)$ and $s_2(t)$ on the segment $[0,T]$. And from expression $(5')$ it follows that the functions $v_1(x,t)$ and $v_2(x,t)$ also coincide. The theorem is proved.

References